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Solutions controllable in small finite and infinitesimal theories of elastic dielectrics

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SUMMARY

An electric field and a deformation constitute a controllable state if they can be maintained in every homogeneous, isotropic, elastic dielectric without the body force and distributed charge. The controllable states possible for small finite theories of compressible elastic dielectrics are determined. Also obtained are the controllable states of the classical electrostriction theory.

1. Introduction

A controllable state is defined as one in which a deformation mapping and an electric field are prescribed at the outset, and then it is shown that such a combination satisfies all the constitutive and field equations without the body force or distributed charge in every homogeneous, isotropic, elastic dielectric. The surface tractions and the applied electric field required to maintain such a state are then calculated at the boundary. For instance, a pure homogeneous deformation interacted with a homogeneous non-zero electric field always constitutes a controllable state because constitutive equations furnish constant stresses and constant dielectric displacement field which will satisfy the field equations identically irrespective of the nature of response coefficients characterizing the material.

When the form of the stored energy function is purely arbitrary, there is a large number of controllable states for homogeneous, isotropic, incompressible elastic dielectrics [1] involving interaction of non-homogeneous deformations with non uniform electric fields. However, Singh [2] has proved that when the dielectric is compressible, the only possible controllable states are those for which the electric field and the strains are both constant.

Considerable analytical simplification can be gained by using suitable approximations to the stored energy function [3]. The first order finite approximation, which reduces to Mooney form [4] in the absence of electrical effects, can be applied to solve problems in which the principal stretches are small and the electric field strength sufficiently weak. It is interesting to observe that the number of controllable states for incompressible dielectrics is considerably larger with approximate forms of the stored energy function than what one obtains with a purely arbitrary form [3].

In the present work we propose to extend the range of investigation of controllable states for special forms of the stored energy function. In Section 5, we proceed to determine all possible such states for first order small finite theory of homogeneous, *compressible*, isotropic, elastic dielectrics. We prove that only homogeneous states are controllable. Therefore the complete class of controllable solutions for the first order finite theory is equivalent to that for the general theory. This, however, is not the case when the dielectric is incompressible [3].

The latter part of this paper is devoted to finding controllable states in the infinitesimal theory of elastic dielectrics, known as classical linear electrostriction [5], [6]. Without the electrical effects, the controllable deformations of the infinitesimal theory of elasticity are given by the displacement fields which are harmonic and whose divergence is uniform [7]. We have demonstrated in Sections 6 and 7 that when such displacement fields are interacted with uniform electric fields, they constitute states that are controllable in coupled as well as uncoupled theories of electrostriction. Furthermore, we prove that such states are the only ones controllable.

Since a controllable state does not require beforehand the knowledge of the functional form of the stored energy function, a comparison of theoretical and experimental results could be used to determine the response coefficients itself for the dielectric considered. Whereas this important feature of controllable deformations has been utilized extensively in finite elasticity [8], the experimentation of similar nature has not been attempted for elastic dielectrics so far.

2. Continuum electroelastostatics

We consider a continuous, homogeneous, isotropic, elastic dielectric solid. The continuum is deformed and polarized by applied mechanical forces and an applied electric field. The deformation mapping is described by

$$x_i = x_i(X_1, X_2, X_3)\,, \qquad i = 1, 2, 3\,, \tag{2.1}$$

where x_i and X_A denote, respectively, the coordinates of a generic particle in the deformed and undeformed configurations, both referred to a fixed rectangular Cartesian system.

When there is no distributed charge, the field equations valid both inside and outside the dielectric are [3]:

$$E_{i,j} = E_{j,i}\,, \tag{2.2}$$

$$D_{i,i} = 0\,, \tag{2.3}$$

$$\sigma_{ij,j} = 0\,, \tag{2.4}$$

and

$$\sigma_{ij} = \sigma_{ji}\,. \tag{2.5}$$

Constitutive relations for free space are:

$$D_i = \epsilon E_i\,, \tag{2.6}$$

$$\sigma_{ij} = M_{ij} = \epsilon E_i E_j - \frac{\epsilon}{2} E_k E_k \delta_{ij}\,, \tag{2.7}$$

whereas inside the dielectric body,

$$D_i = \frac{2\rho_0}{I_6^{\frac{1}{2}}}\left\{\frac{\partial W}{\partial I_4}\delta_{ij} + \frac{\partial W}{\partial I_5} g_{ij} + \frac{\partial W}{\partial I_6} g_{ij}^2\right\} E_j\,, \tag{2.8}$$

$$\begin{aligned}\sigma_{ij} = \frac{2\rho_0}{I_6^{\frac{1}{2}}}\Bigg\{&\left[\frac{\partial W}{\partial I_1} + I_1\frac{\partial W}{\partial I_2}\right] g_{ij} - \frac{\partial W}{\partial I_2} g_{ij}^2 \\ &+ I_3\frac{\partial W}{\partial I_3} g_{ij} + \frac{\partial W}{\partial I_4} E_i E_j + \frac{\partial W}{\partial I_5}\left[g_{ik}E_kE_j + g_{jk}E_kE_i\right] \\ &+ \frac{\partial W}{\partial I_6}\left[g_{ik}^2 E_k E_j + g_{jk}^2 E_k E_i + g_{ik} g_{jl} E_k E_l\right]\Bigg\}\,.\end{aligned} \tag{2.9}$$

Here g_{ij} denotes the Finger strain tensor

$$g_{ij} = \frac{\partial x_i}{\partial X_A}\frac{\partial x_j}{\partial X_A}\,, \tag{2.10}$$

and the strain energy function W of the elastic dielectric depends upon the invariants:

$$\begin{aligned}&I_1 = g_{ii}\,,\ I_2 = g_{ii}^2\,,\ I_3 = E_i E_i\,,\\ &I_4 = g_{ij}E_iE_j\,,\ I_5 = g_{ij}^2 E_i E_j\,,\ I_6 = \det g_{ij}\,.\end{aligned} \tag{2.11}$$

In the above equations, E_i represents the electric field strength, D_i the dielectric displacement field, σ_{ij} the stress tensor which accounts for all electromechanical effects except the gravitational and inertial body forces that we shall set to zero in the work to follow. The constant ε is the dielectric constant of free space, ρ_0 the mass density in the undeformed state, and M_{ij} denotes the Maxwell stress tensor present everywhere outside the dielectric.

3. Approximate theories

Assuming the stored energy function $W(I_1, I_2, \ldots, I_6)$ as a polynomial in its arguments, we can approximate W to any desired order in the principal extensions and powers of the electric field by neglecting terms above an appropriate degree in the polynomial expansion [3].

3.1. *First approximation*

It is obtained by neglecting in the polynomial expansion of W terms involving powers higher than second in principal extensions and the electric field components. Constitutive equations (2.8) and (2.9) then take the form [3]:

$$D_i = 2\rho[(a_3-a_4)\delta_{ij}+a_5 J_1 \delta_{ij}+a_4 g_{ij}]E_j\,, \tag{3.1}$$

and

$$\sigma_{ij} = 2\rho\{(a_1+(a_1+2a_2)J_1+a_5 I_4]g_{ij}-a_1 g_{ij}^2 +(a_3-a_4)E_iE_j+a_4[g_{ik}E_kE_j+g_{jk}E_kE_i]\}\,, \tag{3.2}$$

where ρ denotes the mass density of the dielectric in the deformed state, $a_1, a_2, \ldots, a_5$ are material constants, and where $J_1 = I_1 - 3$.

3.2. *Classical coupled theory of electrostriction*

The constitutive equations governing such a theory [3] are:

$$D_i = \rho_0(b_1\delta_{ij}+\tfrac{1}{2}b_1 e_{kk}\delta_{ij}+b_2 e_{ij})E_j\,, \tag{3.3}$$

$$\sigma_{ij} = \rho_0[c_1 e_{kk}\delta_{ij}+c_2 e_{ij}+b_1 E_kE_k\delta_{ij}+\tfrac{1}{2}(b_1+b_2)E_iE_j]\,, \tag{3.4}$$

where $e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$, and where b_1, b_2, c_1, c_2 are all material constants. These constitutive equations can be derived by neglecting in the polynomial expansion for W terms containing powers higher than second in the displacement gradients $\partial u_i/\partial x_j$ and the electric field components E_i.

3.3. *Classical uncoupled theory of electrostriction*

This theory, which is used more than often in literature in solving boundary value problems that arise in electroelasticity [5], [6], is obtained by neglecting the coupling terms $e_{kk}E_i$ and $e_{ij}E_j$ in Eqn. (3.3) to yield:

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij}+2\mu e_{ij}+aE_kE_k\delta_{ij}+bE_iE_j \tag{3.5}$$

$$D_i = kE_i\,, \tag{3.6}$$

where λ, μ, a, b and $k=\rho_0^{-1}(2b-a)$ are material constants.

4. Controllable states

We first select the theory of which the controllable states are desired. Suppose we are given a positive definite matrix g_{ij} which satisfies compatibility conditions of strain and an electric field which is conservative. Using the constitutive equations of the theory selected, we find the corresponding stress σ_{ij} and the dielectric displacement D_i. If it so happens that σ_{ij} so obtained satisfy the stress equation of equilibrium $\sigma_{ij,j}=0$, and D_i meet the flux field equation $D_{i,i}=0$ independent of the material constants involved in the constitutive equations, then the state represented by the strain matrix g_{ij} and the electric field E_i is called a controllable state. The importance of such states is that they can be supported without the body force or distributed charge in every homogeneous, isotropic, elastic dielectric obeying the theory selected.

It is the purpose of this presentation to find all possible controllable states of first order finite theory as well as those of classical uncoupled and coupled theories of electrostriction described in Section 3. We follow the procedure developed by Ericksen [9] and Shield [10].

5. Controllable states of first order finite deformation theory

To seek restrictions on possible g_{ij} and E_i, we substitute from constitutive relation (3.1) and (3.2) into the field equations (2.3) and (2.4):

$$a_1\left(\frac{g_{ij}+J_1 g_{ij}-g_{ij}^2}{J}\right)_{,j} + 2a_2\left(\frac{J_1}{J}g_{ij}\right)_{,j} + a_3\left(\frac{E_iE_j}{J}\right)_{,j}$$

$$+ a_4\left(\frac{g_{ik}E_kE_j+g_{jk}E_kE_i-E_iE_j}{J}\right)_{,j} + a_5\left(\frac{I_4 g_{ij}}{J}\right)_{,j} = 0\,, \tag{5.1}$$

and

$$a_3\left(\frac{E_i}{J}\right)_{,i} + a_4\left(\frac{g_{ij}E_j-E_i}{J}\right)_{,i} + a_5\left(\frac{J_1E_i}{J}\right)_{,i} = 0\,, \tag{5.2}$$

where the Jacobian

$$J = \det\left(\frac{\partial x_i}{\partial X_A}\right) = \frac{\rho_0}{\rho}\,.$$

Necessary and sufficient that Eqns. (5.1) and (5.2) be satisfied for any choice of the material constants $a_1, a_2, \ldots, a_5$, the coefficients of each of a's should separately vanish:

$$\left(\frac{g_{ij}-g_{ij}^2}{J}\right)_{,j} = 0\,, \tag{5.3}$$

$$\left(\frac{J_1}{J}g_{ij}\right)_{,j} = 0\,, \tag{5.4}$$

$$\left(\frac{E_iE_j}{J}\right)_{,j} = 0\,, \tag{5.5}$$

$$\left(\frac{g_{ik}E_kE_j+g_{jk}E_kE_i}{J}\right)_{,j} = 0\,, \tag{5.6}$$

$$\left(\frac{I_4 g_{ij}}{J}\right)_{,j} = 0\,, \tag{5.7}$$

$$\left(\frac{E_i}{J}\right)_{,i} = 0\,, \tag{5.8}$$

$$\left(\frac{g_{ij}E_j}{J}\right)_{,i} = 0\,, \tag{5.9}$$

$$\left(\frac{J_1E_i}{J}\right)_{,i} = 0\,. \tag{5.10}$$

Besides, the field E_i has to be conservative:

$$E_{i,j} = E_{j,i}\,. \tag{5.11}$$

And g_{ij} must meet the compatibility conditions:

$$R_{ijkl} = 2\left[g_{il,kj}^{-1}+g_{jk,il}^{-1}-g_{ik,jl}^{-1}-g_{jl,ik}^{-1}\right]+g_{mn}(A_{jkm}A_{iln}-A_{jlm}A_{ikn}) = 0\,, \tag{5.12}$$

where g_{ij}^{-1} denotes the inverse of the matrix g_{ij}, and where

$$A_{ijk} = g_{ik,j}^{-1}+g_{jk,i}^{-1}-g_{ij,k}^{-1}\,.$$

Necessary and sufficient for a positive definite symmetric tensor g_{ij} and the field E_i to combine to form a controllable state is that the conditions (5.3) to (5.12) be all satisfied.

From Eqns. (5.5) and (5.8), we obtain

$$E_{i,j}E_j = 0\,,$$

which in view of Eqn. (5.11) gives

$$(E_j E_j)_{,i} = 0\,. \tag{5.13}$$

Equations (5.13) and (5.7) together yield

$$\left(\frac{1}{J}\,g_{ij}\right)_{,j} = 0\,, \tag{5.14}$$

or

$$\frac{\partial}{\partial x_j}\left(\frac{1}{J}\frac{\partial x_j}{\partial x_A}\right)\frac{\partial x_i}{\partial X_A} + \frac{1}{J}\frac{\partial}{\partial x_j}\left(\frac{\partial x_i}{\partial X_A}\right)\frac{\partial x_j}{\partial X_A} = 0\,.$$

Since $J \neq 0$ and $\dfrac{\partial}{\partial x_j}\left(\dfrac{1}{J}\dfrac{\partial x_j}{\partial X_A}\right)$ vanishes identically, we conclude that

$$x_{i,AA} = 0\,. \tag{5.15}$$

With Eqns. (5.4) and (5.14),

$$\frac{1}{J}\,g_{ij}J_{1,j} = 0\,. \tag{5.16}$$

Since the matrix g_{ij} is positive definite, Eqn. (5.16) yields

$$J_{1,i} = 0$$

or that J_1 and hence I_1 is constant. The Laplacian of I_1 is therefore zero. That is

$$\left(\frac{\partial x_i}{\partial X_B}\frac{\partial x_i}{\partial X_B}\right)_{,AA} = 0\,,$$

or

$$x_{i,AAB}x_{i,B} + x_{i,AB}x_{i,AB} = 0\,. \tag{5.17}$$

In view of Eqn. (5.15), Eqn. (5.17) yields

$$x_{i,AB} = 0\,. \tag{5.18}$$

The functions $x_i(X_A)$ are linear in arguments X_A so that g_{ij} is a constant tensor. It also then follows that J is a constant. Using (5.8) now, we obtain

$$E_{i,i} = 0\,. \tag{5.19}$$

From Eqn. (5.13) with use of Eqns. (5.11) and (5.19), we have

$$\begin{aligned} 0 = (E_j E_j)_{,ii} &= E_{j,ii}E_j + E_{j,i}E_{j,i}\,, \\ &= E_{i,ij}E_j + E_{j,i}E_{j,i}, \\ &= E_{j,i}E_{j,i}\,, \end{aligned}$$

thus implying that E_i is uniform. It is now readily seen that with g_{ij} and E_i both constant, all the conditions (5.3) to (5.12) are identically satisfied. Hence, the only controllable states for first order finite theory of homogeneous, isotropic, compressible elastic dielectrics are homogeneous deformations combined with uniform electric fields.

For any second or higher order finite approximation to W, the controllability conditions on g_{ij} and E_i will obviously include those for the first order approximation. Since homogeneous states are always controllable, it follows that for all finite theories of deformation of compressible dielectrics, including the one where the stored energy function is purely arbitrary, the only controllable states are homogeneous strains accompanied with uniform electric fields.

6. Controllable states of coupled theory of electrostriction

The constitutive equations which define this theory are (3.3) and (3.4). Substitution in the field equations (2.3) and (2.4) gives

$$b_1 E_{i,i} + \tfrac{1}{2} b_1 (e_{kk} E_i)_{,i} + b_2 (e_{ij} E_j)_{,i} = 0\,, \tag{6.1}$$

and

$$c_1 (e_{kk})_{,i} + c_2 (e_{ij})_{,j} + b_1 (E_k E_k)_{,i} + \tfrac{1}{2}(b_1 + b_2)(E_i E_j)_{,j} = 0\,. \tag{6.2}$$

If e_{ij} and E_i are to combine for a controllable state, then Eqns. (6.1) and (6.2) must be satisfied for all values of material constants b_1, b_2, c_1, and c_2. Necessary and sufficient for which is that

$$E_{i,i} + \tfrac{1}{2}(e_{kk} E_i)_{,i} = 0\,, \tag{6.3}$$

$$(e_{ij} E_j)_{,i} = 0\,, \tag{6.4}$$

$$(e_{kk})_{,i} = 0\,, \tag{6.5}$$

$$(e_{ij})_{,j} = 0\,, \tag{6.6}$$

$$(E_k E_k)_{,i} + \tfrac{1}{2}(E_i E_j)_{,j} = 0\,, \tag{6.7}$$

$$(E_i E_j)_{,j} = 0\,. \tag{6.8}$$

In addition, the electric field has to be conservative:

$$E_{i,j} = E_{j,i}\,, \tag{6.9}$$

and the infinitesimal strains e_{ij} must be compatible:

$$e_{ij,kl} + e_{kl,ij} - e_{il,kj} - e_{kj,il} = 0\,. \tag{6.10}$$

Since $|e_{ij}| \ll 1$, Eqns. (6.3) and (6.5) give

$$E_{i,i} = 0\,. \tag{6.11}$$

Equations (6.7) and (6.8) furnish

$$(E_k E_k)_{,i} = 0\,. \tag{6.12}$$

Equations (6.9), (6.11), and (6.12) require that the field E_i must be uniform. The restrictions on strains e_{ij} are then only Eqns. (6.5), (6.6), and (6.10). In terms of the displacement field $\boldsymbol{u}$, these conditions become

$$\nabla^2 \boldsymbol{u} = 0\,, \tag{6.13}$$

and

$$\nabla(\nabla \cdot \boldsymbol{u}) = 0\,. \tag{6.14}$$

Thus any displacement field given by Eqns. (6.13) and (6.14) when coupled with any uniform electric field furnishes a controllable state and such states are the only ones controllable.

7. Controllable states of uncoupled theory of electrostriction

Constitutive equations that govern this theory are Eqns. (3.5) and (3.6), which when substituted in Eqns. (2.3) and (2.4) yield

$$k E_{i,i} = 0\,, \tag{7.1}$$

and

$$\lambda (e_{kk})_{,i} + 2\mu (e_{ij})_{,j} + a(E_k E_k)_{,i} + b(E_i E_j)_{,j} = 0\,. \tag{7.2}$$

If Eqns. (7.1) and (7.2) are to hold for all possible values of k, λ, μ, a, and b, then

$$E_{i,i} = 0\,, \tag{7.3}$$

$$(e_{kk})_{,i} = 0\,, \tag{7.4}$$

$$(e_{ij})_{,j} = 0\,, \tag{7.5}$$

$$(E_k E_k)_{,i} = 0\,, \tag{7.6}$$

$$(E_i E_j)_{,j} = 0\,. \tag{7.7}$$

Conditions (7.3), (7.6), and that E_i has to be conservative, require the field E_i to be uniform.

The displacement field $\boldsymbol{u}$ for a controllable state is once again given by Eqns. (7.4) and (7.5), so that

$$\nabla^2 \boldsymbol{u} = 0\,, \tag{7.8}$$

and

$$\nabla (\nabla \cdot \boldsymbol{u}) = 0\,. \tag{7.9}$$

It may be remarked here that when the electric field is absent, the displacement fields for controllable states in the infinitesimal theory of elasticity [7] are indeed given by Eqns. (7.8) and (7.9). What we have shown in Sections 6 and 7 is that in the coupled as well as uncoupled theories of electrostriction, the displacement fields satisfying Eqns. (7.8) and (7.9) simultaneously are still controllable when combined with uniform electric fields, and that states of this type are the only ones controllable.

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